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Shear layer instability of an inviscid compressible fluid. Part 3

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This note corrects and augments the results of Blumen, Drazin & Billings (1975) on the linear instability of a shear layer between two streams of an inviscid compressible fluid at the same temperature. The main conclusion of these authors, that the shear layer is unstable to two-dimensional disturbances however large the Mach number, is confirmed in the present paper. Also a third mode of instability is discovered and the character of all three modes revealed by a series of accurate calculations.

The instability of a shear layer of an inviscid compressible fluid is a classical problem of fluid mechanics, which has attracted the attention of some distinguished scientists of earlier generations, notably Ackeret (1928) and Landau; and for this reason alone it is desirable to set the record straight. Although the work below is very specialized, it does, however, have a wider significance both as a case study of the great care necessary to unravel problems of instability when there are several modes, and as a prototype of similar problems of instability of plane parallel flow of inviscid fluid under the influence of some other external force field, for example buoyancy or variable Coriolis parameter.

Following Blumen (1970) we consider the instability of a basic shear layer, with velocity $\mathbf{u} = \tanh y\,\mathbf{i}$, of an unbounded inviscid perfect gas at uniform temperature. For this problem Blumen (1970) found only one unstable mode, which is stationary, and Blumen *et al.* (1975) discovered a new pair of unstable modes with equal but opposite phase velocities. This note shows that the pattern of unstable modes is even more complicated than as described by Blumen *et al.* (1975), who confused the modes where the value of the Mach number M (based on half the velocity difference across the shear layer) is quite close to unity. However, their general physical conclusions, their asymptotic results (in particular their equation (14), about which they expressed doubt), and most of their numerical results are confirmed.

We follow the notation and the statement of the problem of these authors and, in particular, we suppose that each normal mode varies like $\exp\{i\alpha(x-ct)\}$, where $c = c_r + ic_i$, so that the relative growth rate of a mode is αc_i and its real phase speed is c_r. We shall *not* be concerned herein with the three *continuous* spectra of the problem which would seem to occur with real c and $-1 < c < 1$, $M^{-2} < (1-c)^2$ and $M^{-2} < (1+c)^2$.

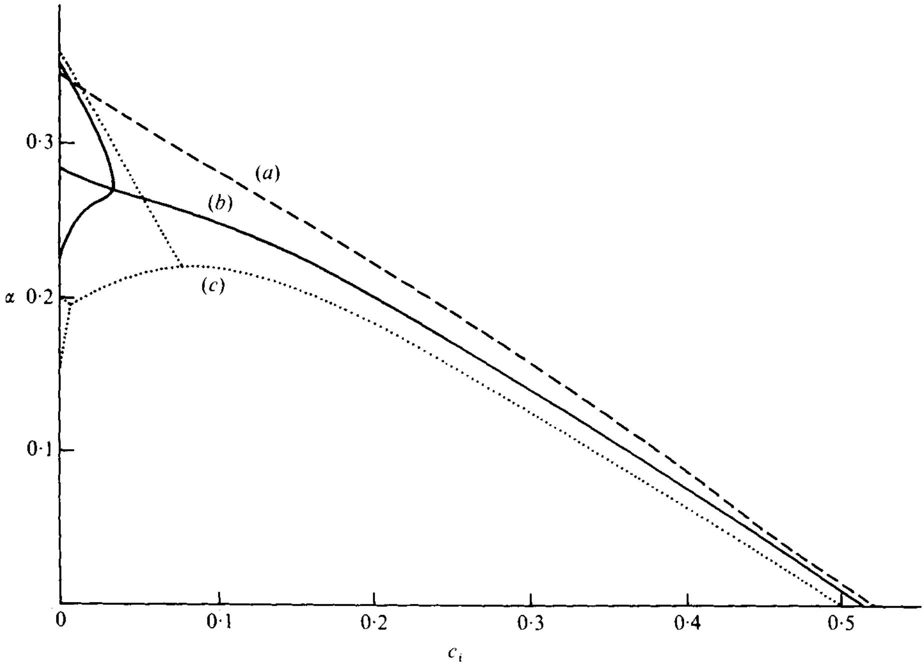

FIGURE 1. Graphs of c_i against α. (*a*) The curve for $M = 0{\cdot}94$, denoted by a broken line. $c_r = 0$. (*b*) The curve for $M = 0{\cdot}96$, denoted by a continuous line. $c_r \neq 0$ on the upper- and lowermost of the nearly straight parts with ends on the α axis; $c_r = 0$ on the rest. (*c*) The curve for $M = 0{\cdot}98$, denoted by a dotted line. $c_r \neq 0$ on the segment whose ends are the upper- and lowermost points of the graph on the α axis; $c_r = 0$ on the rest of the graph.

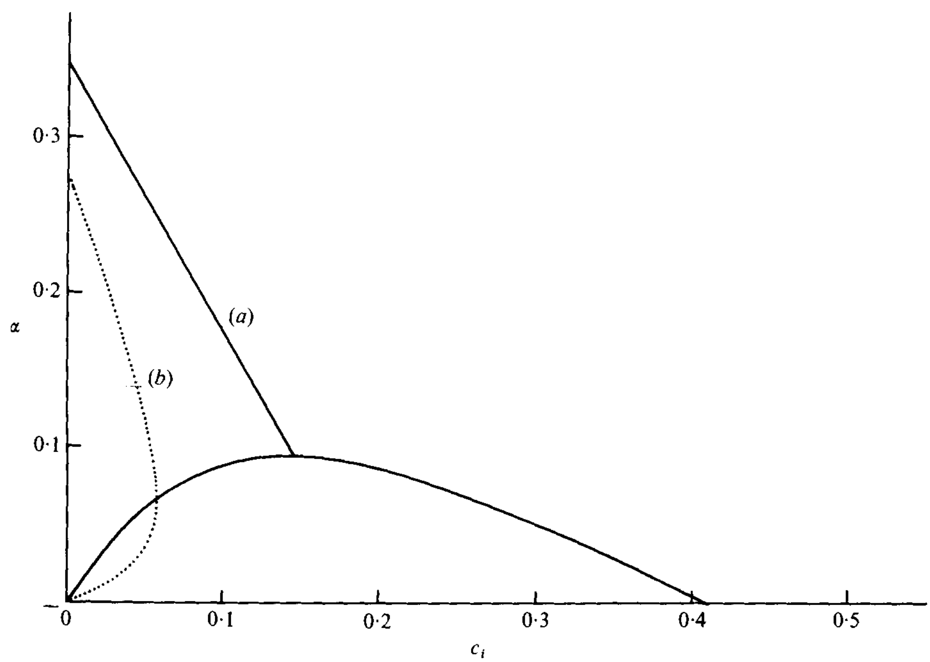

FIGURE 2. Graphs of c_i against α. (*a*) The curve for $M = 1{\cdot}1$, denoted by a continuous line. $c_r \neq 0$ on the nearly straight part; $c_r = 0$ on the rest. (*b*) The curve for $M = 1{\cdot}5$, denoted by a dotted line. $c_r \neq 0$.

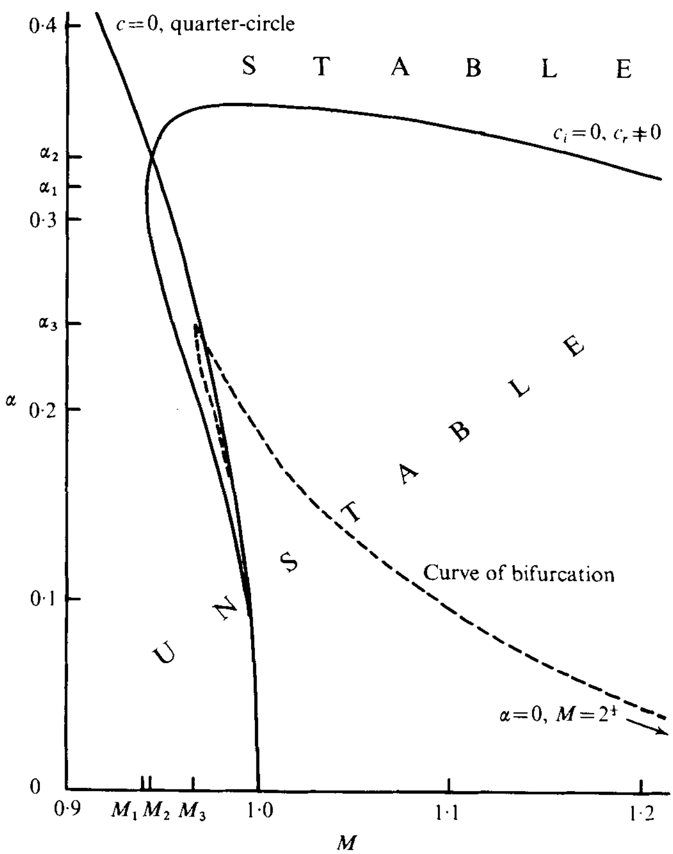

FIGURE 3. The M, α plane near $M = 1$. Continuous lines denote marginal curves for some mode ($c = 0$) or modes ($c_r \neq 0$, $c_i = 0$) and the broken line denotes the curve of bifurcation.

It seems simplest to describe our new results graphically, so we shall just present some figures and explain them briefly. Figures 1 and 2 show graphs of the imaginary part c_i of the phase speed of the unstable modes as a function of the wavenumber α for five typical values of the Mach number. For each mode with $c_r > 0$ there exists a similar mode (essentially the mode reflected in the origin of the x, y plane) with the same values of M, α and c_i but with opposite phase speed $-c_r$; so a curve in the figures denoted by '$c_r \neq 0$' represents *two* travelling modes with phase speed $\pm c_r$. We have chosen typical values of M so that the figures illustrate the main topological types of graph. These figures can be understood better with use of figure 3, which depicts the M, α plane near $M = 1$.

Thus curve (a) in figure 1 is typical of graphs for $0 \leqslant M < M_1$, say, where we computed $M_1 = 0{\cdot}9413$. For this case there is a unique unstable mode, which is stationary, so that $c_r = 0$. As M increases through the value M_1 there appears a pair of similar modes, first with $c_r = \pm 0{\cdot}0905$ at the point $c_i = 0$, $\alpha = \alpha_1 = 0{\cdot}317$ and then as a segment. As M increases above M_1 the segment representing the pair of travelling modes grows and approaches the curve $c_r = 0$. The segment meets the curve $c_r = 0$ at $c_i = 0$, $\alpha = \alpha_2$ when $M = M_2$, say, where we computed $M_2 = 0{\cdot}9430$ and $\alpha_2 = 0{\cdot}333$. Then as M increases above M_2 the segment crosses the curve, as shown in figure 1, curve (b), which is typical of the case $M_2 < M < M_3$. At the intersection the segment represents two travelling modes $\pm c_r \neq 0$ and the curve one stationary mode $c_r = 0$ independently until M attains the value M_3.

At this value of M the values of $\pm c_r$ at the intersection of the segment and the curve become zero and the modes bifurcate. We computed this first point of bifurcation as $M_3 = 0{\cdot}9674$, $\alpha_3 = 0{\cdot}245$. The typical case for $M_3 < M < 1$ is illustrated by curve (c) in figure 1. We see that there are either none, one or three unstable modes for each value of α. (Remember that '$c_r \neq 0$' denotes two unstable travelling modes.) As $M \uparrow 1$ one of the bifurcation points tends to the origin of the c_i, α plane, there being a cusp at the origin when $M = 1$. For the case when $1 < M < 2^{\frac{1}{2}}$, illustrated by curve (a) in figure 2, the curves are topologically equivalent to the curves for $M = 1$, but the branch enters the origin at a positive angle to the α axis. All the while that M has been increasing, the value of c_i corresponding to $\alpha = 0$ has been decreasing; it reaches zero when $M = 2^{\frac{1}{2}}$. Hence the typical curve for $M > 2^{\frac{1}{2}}$ is as shown by curve (b) in figure 2. As M increases above $2^{\frac{1}{2}}$, the curve shrinks indefinitely towards the origin.

This description can be supplemented with some asymptotic results of Blumen *et al.* (1975). Their formula (14) always gives the unstable stationary mode just inside the circle $\alpha^2 + M^2 = 1$, and their formulae (22) and (23) give the curves as they approach the origin in the c_i, α plane for $1 < M \leqslant 2^{\frac{1}{2}}$. In particular, (23) yields $3c^3 \sim i\alpha$ as $\alpha \downarrow 0$ for $M = 2^{\frac{1}{2}}$, in fair agreement with our numerical results (which are especially difficult to derive accurately when α is small because of the outer boundary condition). We have also used our numerical results for $M = 5$ and 10 and Landau's analytic result for $\alpha = 0$ to suggest that the unstable modes have complex phase speeds given by

$$c = \pm(1 - M^{-1}) + \{\pm(1{\cdot}8\eta^2 + 0{\cdot}75\eta - 0{\cdot}125) + 0{\cdot}28\eta(0{\cdot}42 - \eta)\,i\}\,M^{-3} + O(M^{-4}) \quad \text{as} \quad M \to \infty \qquad (1)$$

for fixed $\eta = \alpha M < 0{\cdot}42$. It should be emphasized that this suggestion is based on a rough fit of our results; it seems, however, to summarize aptly the behaviour of the eigenvalues c for large values of M.

We have written of the bifurcations, and illustrated them in figures 1 and 2. These bifurcations seem to be of the simplest, namely the square-root, kind as recognized and interpreted by Gaster (1968) in a similar problem of hydrodynamic instability. Thus if there is one solution with eigenvalue c_0 and eigenfunction p_0 at a point (M_0, α_0) of bifurcation then there are two neighbouring eigenvalues $c_0 + \delta c$ at points $(M_0 + \delta M, \alpha_0 + \delta\alpha)$ such that

$$\delta c \sim \pm(a_0\,\delta\alpha + b_0\,\delta M)^{\frac{1}{2}} \quad \text{as} \quad \delta\alpha,\, \delta M \to 0, \qquad (2)$$

for some constants a_0 and b_0. This is consistent with our examination of some details, as well as of the qualitative character, of our numerical results. Moreover, applying the analytic method of Banks & Drazin (1973, § 6) to perturb a known eigensolution, one can easily confirm this behaviour, showing that bifurcations may occur where the integral

$$I_0 \equiv 2\int_{-\infty}^{\infty} (U - c_0)^{-3}\,(p_0'^2 + \alpha^2 p_0^2)\,dy$$

vanishes. It then follows that the tangent to the curve of bifurcation at the point (M_0, α_0) is given by

$$I_1 \equiv \int_{-\infty}^{\infty} \{\alpha_0^2\,\delta M^2 + \delta\alpha^2\,[M_0^2 - (U-c_0)^{-2}]\} p_0^2\,dy = 0,$$

i.e. by

$$\delta\alpha \int_{-\infty}^{\infty} \{(\tanh y - c_0)^{-2} - M_0^2\} p_0^2\,dy \sim \alpha_0 M_0\,\delta M \int_{-\infty}^{\infty} p_0^2\,dy. \tag{3}$$

Computation of I_0 at a point of the curve of bifurcation showed that I_0 does indeed vanish there.

These asymptotic results assist interpretation of our calculations of the curve of bifurcation, shown in figure 3. This figure shows how close together are the two marginal curves as they descend to the point $M = 1$, $\alpha = 0$, the curve of bifurcation being narrowly confined by the marginal curve $c_r \neq 0$ on its left and the quarter-circle $c = 0$ on its right. This closeness of the three curves indicates the severe difficulty of numerical calculation of the eigenvalues. To relate figures 1 and 2 to figure 3 it may help to note that the unstable travelling modes are represented in figure 3 by points confined within the marginal curve $c_r \neq 0$ and the unstable stationary modes by points within the quarter-circle or the curve of bifurcation.

At many stages of the computations it was necessary to calculate quite accurately in order to obtain the above results because of the closeness of the eigenvalues and because of the danger of 'mode-jumping' as the parameters α and M were varied.

We used a Runge–Kutta method to integrate the differential equation (equation (2) of Blumen *et al.* 1975) along a contour which was indented below the critical point in the complex y plane and obtained the eigenvalue c by 'shooting'; initially it was necessary to use invariant imbedding by varying c_i, having fixed $c_r = 0$ and chosen appropriate values of M and α, to determine that there are three *and only three* unstable stationary modes.

After submission of the above work, the valuable numerical results of Gropengiesser (1969) were brought to our attention. He took the profile of a free boundary layer between two streams at various different temperatures, rather than of a hyperbolic tangent between streams at the same temperature. He then computed the characteristics of spatially, rather than of temporally, growing modes. So his results are not strictly comparable with ours. Further, his results were computed only for $0 \leqslant M \leqslant \frac{3}{2}$ (based on our definition of M). By judgement of his figures, his numerical results seem to be about as accurate as those of Blumen *et al.* (1975), but lack analytic results for support and interpretation. Nevertheless Gropengiesser recognized the existence of two unstable modes and made extensive computations. Our more accurate results and qualitative view of the three unstable modes over a larger range of Mach number supplement and interpret his results, which cover a greater variety of shear layers.

We need not elaborate further the details of this problem, but the occurrence of extra modes of instability and the associated numerical difficulty deserve wider attention. It seems that when a basic state permits propagation of a wave, for example sound, the instability of a shear flow may be complicated by the occurrence of extra modes of instability. In extreme ranges of the parameters these modes may be recognized separately as inertial instability or as waves, but in general they are not separable in any simple way. Blumen *et al.* (1975, p. 306) have pointed out some examples of this occurrence of multiple modes. In solving a problem of baroclinic instability, Garcia & Norscini (1970) seem to have revealed another. The literature of the instability of shear flows is scattered with omissions and misinterpretations because of ignorance of extra modes or because of the great difficulty of calculating their characteristics even when their existence is recognized. To take yet another example, Gage & Miller (1974) took great care to calculate the stability characteristics of a jet in a stratified viscous incompressible fluid and recognized 'a curious bimodal behaviour of the curve of neutral stability'. It seems that because of the notorious difficulty of calculating the characteristics of long waves they did not clearly identify the presence of a second mode, which was subsequently revealed by the analytic and numerical work of Silcock (1975).

REFERENCES

ACKERET, J. 1928 Über Luftkrafte bei sehr grossen beschwindigkeiten insbesondere bei ebenen Strömungen. *Helv. Phys. Acta* **1**, 301–322.

BANKS, W. H. H. & DRAZIN, P. G. 1973 Perturbation methods in boundary-layer theory. *J. Fluid Mech.* **58**, 763–775.

BLUMEN, W. 1970 Shear layer instability of an inviscid compressible fluid. *J. Fluid Mech.* **40**, 769–781.

BLUMEN, W., DRAZIN, P. G. & BILLINGS, D. F. 1975 Shear layer instability of an inviscid compressible fluid. Part 2. *J. Fluid Mech.* **71**, 305–316.

GAGE, K. S. & MILLER, J. R. 1974 Linear viscous stability analysis of the stratified Bickley jet. *Phys. Fluids* **17**, 883–887.

GARCIA, R. V. & NORSCINI, R. 1970 A contribution to the baroclinic instability problem. *Tellus* **22**, 239–250.

GASTER, M. 1968 Growth of disturbances in both space and time. *Phys. Fluids* **11**, 723–727.

GROPENGIESSER, H. 1969 Beitrag zur Stabilität freier Grenzschichten in kompressiblen Medien. *Dtsche Luft Raumfahrt Rep.* no. 69–25.

SILCOCK, G. 1975 On the stability of parallel stratified shear flows. Ph.D. dissertation, University of Bristol.